MATH 3235 Probability Theory

10/27/22

X r.v.

$$m_k(X) = \mathbb{E}(X^k) = \begin{cases} \int_{\mathbb{R}} x^k f_X(x)\, dx \\ \sum_x x^k p_X(x) \end{cases}$$

$m_0(X) = 1$

$m_1(X) = \mathbb{E}(X)$ $\quad m_2(X) = \mathbb{E}(X^2)$

$\mathrm{var}(X) = m_2(X) - m_1(X)^2$

$a, b \in \mathbb{R}$ $\quad X_1 \quad X_2$ r.v.

$m_1(aX_1 + bX_2) = a\, m_1(X_1) + b\, m_1(X_2)$

$X_1 \quad X_2 \qquad f(x_1, x_2)$

$$\mathbb{E}(X_1^2) = \iint x_1^2 f(x_1, x_2)\, dx_1\, dx_2$$

$$= \int x_1^2 f_{X_1}(x_1)\, dx_1$$

if $X_2 = 1$ with prob 1

$$m_1(aX + b) = a\, m_1(X) + b$$

$$\mathrm{var}(aX + b) = a^2\, \mathrm{var}(X)$$

$$\sigma_{aX+b} = |a|\, \sigma_X$$

$$\mathrm{var}(X_1 + X_2) = \mathrm{var}(X_1) + \mathrm{var}(X_2) + 2\,\mathrm{cov}(X_1, X_2)$$

$$\mathrm{var}(aX_1 + bX_2) = a^2\,\mathrm{var}(X_1) + b^2\,\mathrm{var}(X_2) + 2ab\,\mathrm{cov}(X_1, X_2)$$

if $X_1 \perp\!\!\!\perp X_2$

$$\mathrm{var}(X_1 + X_2) = \mathrm{var}(X_1) + \mathrm{var}(X_2)$$

$$\mathrm{var}(X) = \mathbb{E}\left((X - \mathbb{E}(X))^2\right) =$$

$$\mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$\mathrm{var}(X) \geq 0 \implies \mathbb{E}(X^2) \geq \mathbb{E}(X)^2$$

Exp r.v. par λ

$$\lambda e^{-\lambda x}$$

$$m_k(X) = \int_0^\infty \lambda x^k e^{-\lambda x} dx$$

$$y = \lambda x$$

$$= \int_0^\infty \lambda (x\lambda)^k \lambda^{-k} e^{-\lambda x} \frac{d(\lambda x)}{\lambda} =$$

$$= \lambda^{-k} \int_0^\infty y^k e^{-y} dy$$

$$\frac{d}{dy} e^{-y} = -e^{-y}$$

$$= -y^k e^{-y}\Big|_0^\infty + \int_0^\infty k y^{k-1} e^{-y}\, dy$$

$$(FG)' = F'G + FG'$$

$$\int F'G = FG - \int FG'$$

$$\int y^k e^{-y}\, dy = k \int y^{k-1} e^{-y}\, dy$$

$$\int e^{-y}\, dy = 1$$

$$\int y e^{-y}\, dy = 1$$

$$\int y^2 e^{-y}\, dy = 2 \cdot 1 = 2$$

$$\int y^3 e^{-y}\, dy = 3 \cdot 2 \cdot 1 = 6$$

$$\int y^k e^{-y}\, dy = k!$$

Induction

$$a_k = \int_0^\infty y^k e^{-y} \, dy = k!$$

$a_0 = 1$ works for $k = 0$

$$a_{k+1} = \int_0^\infty y^{k+1} e^{-y} \, dy = (k+1) \, a_k =$$

$$= (k+1) \, k! = (k+1)!$$

X m_k

X discrete with value $\mathbb{N}$

$$p_k = \mathbb{P}(X = k)$$

$$G_X(\lambda) = \sum_k \lambda^k p_k = \mathbb{E}(\lambda^X)$$

$$M_X(t) = \sum_{k=0}^\infty \frac{t^k}{k!} m_k = \mathbb{E}(e^{tX})$$

$$\mathbb{E}(e^{tX}) = \mathbb{E}\left(\sum_{k=0}^{\infty} \frac{t^k X^k}{k!}\right) =$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbb{E}(X^k)$$

$$M_X(t) = \mathbb{E}(e^{tX})$$

P.g.f $G_X(\lambda) \to p_k = \mathbb{P}(X = k)$

p_k define X uniquely.

X is any r.v.

if I have all the moments

m_k of X $\Longrightarrow$ I know the

p.d.f. of X?

If X_1 and X_2 are r.v. and

$m_k(X_1) = m_k(X_2)$ $\forall k$ then

The p.d.f of X_1 is equal to The p.d.f. of X_2.

No! In general.

But

$$\sum_k \frac{t^k}{k!}|m_k| < +\infty \quad \text{for}$$

some t, Then uniqueness holds.

$$h(x) = \begin{cases} 0 & x < 0 \\ e^{-\frac{1}{x}} & x > 0 \end{cases}$$

$$\frac{d^k}{dx^k} h(0) = 0 \qquad \forall k$$

$$\frac{1}{x^2} e^{-\frac{1}{x}}$$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$$

$t > 0 \qquad e^{tx}$

$$M_X(0) = 1$$

$M_X(t) < +\infty \qquad t > 0$

iff $\quad f_X(x) \to 0$ exponentially fast when $x \to \infty$

$M_X(t) < +\infty \qquad t < 0$

iff $\quad f_X(x) \to 0$ exponentially fast when $x \to -\infty$

$$f(x) = \frac{1}{\pi} \frac{1}{1 + x^2} \qquad \text{Cauchy}$$

$$M_X(t) = \begin{cases} 1 & x = 0 \\ +\infty & x \neq 0 \end{cases}$$

$$G_X(\lambda) = \mathbb{E}(\lambda^X) \qquad \lambda = e^t$$

$$G_X(e^t) = \mathbb{E}(e^{tX}) = M_X(t)$$

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{itx} f_X(x)\, dx$$

Charecteristic function of X.

Fourier Transform of f_X

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1) $$\mathbb{E}(X^k) = M_X^{(k)}(0)$$

2) $$M_{X_1+X_2}(t) = \mathbb{E}\left(e^{t(X_1+X_2)}\right) =$$

$$\underline{X_1 \perp\!\!\!\perp X_2} \qquad = \mathbb{E}(e^{tX_1})\,\mathbb{E}(e^{tX_2})$$

$$= M_{X_1}(t)\, M_{X_2}(t)$$

$$\mathbb{E}\left(e^{t(X_1+X_2)}\right) = \mathbb{E}\left(e^{tX_1} e^{tX_2}\right) =$$

$$\mathbb{E}\left(e^{tX_1}\right)\mathbb{E}\left(e^{tX_2}\right)$$

if $X_1 \perp\!\!\!\perp X_2$

3) if $M_X(t)$ exists for $T \neq 0$

Then $M_X(t)$ defines uniquely

The p.d.f. of X.

X is exponential r.v.

$$M_X(t) = \lambda \int_0^\infty e^{tx} e^{-\lambda x}\, dx =$$

$$= \lambda \int_0^\infty e^{(t-\lambda)x}\, dx =$$

$$= \frac{\lambda}{t-\lambda} e^{(t-\lambda)x} \Big|_0^\infty =$$

$$= \frac{\lambda}{t - \lambda} \left(e^{(t-\lambda)\infty} - 1 \right)$$

$$M_X(t) = \begin{cases} \dfrac{\lambda}{\lambda - t} & \text{if } t < \lambda \\ +\infty & \text{if } t \geq \lambda \end{cases}$$